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Thermal State for the Capacitance Coupled Mesoscopic Circuit with a Power Source

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The Schrödinger equation of the mesoscopic capacitance coupled circuit with an arbitrary power source is solved by means of two step unitary transformation. The original Hamiltonian transformed to a very simple form by unitary operators so that it can be easily treated. We derived the exact full wave functions in Fock state. By making use of these wave functions and introducing the Lewis–Riesenfeld invariant operator, the thermal state have been constructed. The fluctuations of charges and currents are evaluated in thermal state. For $T \to 0$, the uncertainty products between charges and currents in thermal state recovers exactly to that of Fock state with $n, m = 0$.

KEY WORDS: mesoscopic capacitance coupled circuit; thermal state; uncertainty product.

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1. INTRODUCTION

The quantum mechanical problem of the time-dependent Hamiltonian systems (TDHS) have been interested in the literature since the introduction of invariant operator by Lewis (1967). The exact quantization of TDHS become possible using several techniques such as propagator method (Gweon and Choi, 2003; Yeon *et al.*, 1993, 1996), invariant operator method (Choi, 2003, 2004; Choi and Gweon, 2003; Um *et al.*, 1997) and unitary(or canonical) transformation method (Choi *et al.*, 2002; Ji and Kim, 1996; Landovitz *et al.*, 1979; Zhang *et al.*, 2001, 2002a,b). We will use unitary transformation method in order to derive quantum mechanical solution of mesoscopic capacitance coupled circuit with a time-dependent power source. Although many actual dynamical systems have been solved using approximation techniques and/or perturbation theory, we will investigate the exact solution of the Schrödinger equation of the system.

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The development of mesoscopic physics and nanoelectronics have been especially rapid during the last two decades due to the advance of lithography techniques and crystal growth. The miniaturization of integrated circuits and components towards atomic scale dimensions required the development of quantum theory on a mesoscopic circuit, since the charge carriers exhibit quantum mechanical properties while the application of classical mechanics invalid (Buot, 1993). The quantum fluctuations of charge and current for the time-dependent LC circuit (Baseia and De Brito, 1993) and RLC linear circuit (Chen *et al.*, 1995; Louisell, 1973; Zhang *et al.*, 1998) with a power source have been investigated in the literatures. In the previous paper (Choi *et al.*, 2002), using unitary transformation approach, we obtained wave functions with continuous spectrum as well as discrete spectrum for the RLC linear circuit driven by time-dependent electromotive force by introducing classical particular solutions of the system. Recently, quantum properties such as uncertainties of charges and currents for the two dimensional mesoscopic circuits with no power source are studied by Zhang *et al.* (2001, 2002a,b) using canonical and unitary transformation method. These quantum analysis of the electric circuit can also be applied to electrical equivalent circuit for arrangement of trapped ion driven by a signal source attached to the trap end caps (Heinzen and Wineland, 1990; Wineland and Dehmelt, 1975).

The quantum properties of the lossless LC circuit with a power source were firstly investigated by Louisell (1973). However, Louisell's study considered no thermal effect. The main purpose of this paper is to construct the thermal state of the mesoscopic capacitance coupled circuit with a power source. The well known Liouville–von Neumann equation (Isihara, 1971; Robertson, 1993) for the nonequilibrium dynamics can be applicable to both time-dependent harmonic and unharmonic oscillators.

This paper organized as the following order. In Section 2, we derive the time-dependent Hamiltonian of the system from classical equation of motion for charges and, by introducing unitary operators, transform it to a quite simple Hamiltonian whose quantum solution is easily solved. The exact wave function of the system is investigated in Section 3. The thermal states of the system is derived by introducing invariant operator in Section 4. In Section 5, the fluctuations of the canonical variables and uncertainty relations between charges and their conjugate currents will be investigated in the thermal state. Finally, we summarize the results of the previous sections in the last section.

2. UNITARY TRANSFORMATION

We consider two loop of LC circuit coupled via capacitance as in Fig. 1. The application of Kirchhoff's law to each loop leads to the equation of motions for

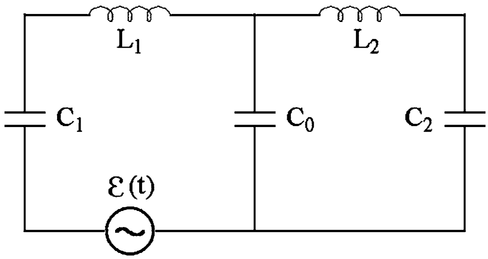

Fig. 1. Capacitance coupled mesoscopic circuit with a power source.

charges:

$$L_1\frac{d^2q_1}{dt^2}+\frac{q_1}{C_1}+\frac{q_1-q_2}{C_0}=\varepsilon(t), \tag{1}$$

$$L_2\frac{d^2q_2}{dt^2}+\frac{q_2}{C_2}-\frac{q_1-q_2}{C_0}=0, \tag{2}$$

where q_j(hereafter all j is $j=1,2$) are charges stored in C_j, respectively, and $\varepsilon(t)$ is an arbitrary power source. Note that q_j can be replaced with operators, $\hat{q}_j$, in order to treat the system in view of quantum mechanics. Then, the corresponding Hamiltonian is given by

$$\hat{H}=\frac{\hat{p}_1^2}{2L_1}+\frac{\hat{p}_2^2}{2L_2}+\frac{\hat{q}_1^2}{2C_1}+\frac{\hat{q}_2^2}{2C_2}+\frac{(\hat{q}_1-\hat{q}_2)^2}{2C_0}-\varepsilon(t)\hat{q}_1, \tag{3}$$

where $\hat{p}_j$ are conjugate current operators of $\hat{q}_j$, that are defined by $\hat{p}_j=-i\hbar\partial/(\partial q_j)$. They satisfies commutation relations such as $[\hat{q}_j,\hat{p}_j]=i\hbar$. By applying canonical equations of Hamiltonian, we can easily check that Eq. (3) satisfies Eqs. (1) and (2).

Unitary transformation approach is one of the convenient methods in order to derive the quantum mechanical solution of the TDHS. To convert Eq. (3) into a simple form whose quantum solution is well known, we use two step unitary transformation. As a first step, we introduce a unitary operator that is given by

$$\begin{aligned}\hat{U}_A=&\exp\left[\frac{i}{\hbar}(\hat{p}_1\hat{q}_1+\hat{q}_1\hat{p}_1)\ln\left(\frac{L_1}{L_2}\right)^{1/8}\right]\\&\times\exp\left[\frac{i}{\hbar}(\hat{p}_2\hat{q}_2+\hat{q}_2\hat{p}_2)\ln\left(\frac{L_2}{L_1}\right)^{1/8}\right]\\&\times\exp\left[-\frac{i\varphi}{\hbar}(\hat{p}_1\hat{q}_2-\hat{p}_2\hat{q}_1)\right],\end{aligned} \tag{4}$$

where

$$\varphi = \frac{1}{2}\tan^{-1}\frac{2\sqrt{L_1L_2}}{L_2(1+C_0/C_1) - L_1(1+C_0/C_2)}. \tag{5}$$

Note that the range of φ is restricted to $-\pi/4 \le \varphi \le \pi/4$. The transformation of Eq. (3) by Eq. (4) can be performed as

$$\hat{H}_A = \hat{U}_A^{-1}\hat{H}\hat{U}_A - i\,\hbar\hat{U}_A^{-1}\frac{\partial\hat{U}_A}{\partial t}. \tag{6}$$

Then, after straightforward calculation, the transformed Hamiltonian is given by

$$\hat{H}_A(\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2, t) = \sum_{j=1}^{2}\hat{H}_{Aj}(\hat{q}_j, \hat{p}_j, t), \tag{7}$$

where

$$\hat{H}_{A1}(\hat{q}_1, \hat{p}_1, t) = \frac{\hat{p}_1^2}{2\sqrt{L_1L_2}} + \frac{1}{2}\sqrt{L_1L_2}\omega_1^2\hat{q}_1^2 - \hat{q}_1\varepsilon(t)\left(\frac{L_2}{L_1}\right)^{1/4}\cos\varphi, \tag{8}$$

$$\hat{H}_{A2}(\hat{q}_2, \hat{p}_2, t) = \frac{\hat{p}_2^2}{2\sqrt{L_1L_2}} + \frac{1}{2}\sqrt{L_1L_2}\omega_2^2\hat{q}_2^2 - \hat{q}_2\varepsilon(t)\left(\frac{L_2}{L_1}\right)^{1/4}\sin\varphi. \tag{9}$$

Here ω_j is

$$\omega_j = \sqrt{\frac{\alpha_j}{\sqrt{L_1L_2}}}, \tag{10}$$

with

$$\alpha_1 = \sqrt{\frac{L_2}{L_1}}\left(\frac{1}{C_0}+\frac{1}{C_1}\right)\cos^2\varphi + \sqrt{\frac{L_1}{L_2}}\left(\frac{1}{C_0}+\frac{1}{C_2}\right)\sin^2\varphi + \frac{\sin 2\varphi}{C_0}, \tag{11}$$

$$\alpha_2 = \sqrt{\frac{L_2}{L_1}}\left(\frac{1}{C_0}+\frac{1}{C_1}\right)\sin^2\varphi + \sqrt{\frac{L_1}{L_2}}\left(\frac{1}{C_0}+\frac{1}{C_2}\right)\cos^2\varphi - \frac{\sin 2\varphi}{C_0}. \tag{12}$$

Even though the original Hamiltonian Eq. (3) involves a cross term $\hat{q}_1\hat{q}_2$, Eqs. (8) and (9) contains no such cross term. However, they still involves power source term. We concentrate on eliminating power source terms in the second step of transformation. To do this, another unitary operator may be introduced as

$$\hat{U}_B = \exp\left[\frac{i}{\hbar}(p_{1p}(t)\hat{q}_1 + p_{2p}(t)\hat{q}_2)\right] \times \exp\left[-\frac{i}{\hbar}(q_{1p}(t)\hat{p}_1 + q_{2p}(t)\hat{p}_2)\right]. \tag{13}$$

Here, $q_{jp}(t)$, $p_{jp}(t)$ are classical particular solutions of the firstly transformed system that are described by Eq. (7) with Eqs. (8) and (9) in q and p space, respectively. The intriduction of these classical particular solutions makes easy to solve the problem of electronic circuit coupled to a power source for both classical and quantum viewpoints (Choi, 2006; Choi *et al.*, 2002; Choi and Nahm, in press). Using Hamilton's equations, we can easily confirm that they satisfies the following differential equations

$$\ddot{q}_{1p}(t) + \omega_1^2 q_{1p}(t) - \varepsilon(t)\sqrt[4]{\frac{1}{L_1^3 L_2}}\cos\varphi = 0, \tag{14}$$

$$\ddot{p}_{1p}(t) + \omega_1^2 p_{1p}(t) - \dot{\varepsilon}(t)\sqrt[4]{\frac{L_2}{L_1}}\cos\varphi = 0, \tag{15}$$

$$\ddot{q}_{2p}(t) + \omega_2^2 q_{2p}(t) - \varepsilon(t)\sqrt[4]{\frac{1}{L_1^3 L_2}}\sin\varphi = 0, \tag{16}$$

$$\ddot{p}_{2p}(t) + \omega_2^2 p_{2p}(t) - \dot{\varepsilon}(t)\sqrt[4]{\frac{L_2}{L_1}}\sin\varphi = 0. \tag{17}$$

By the same method as the first step, the second step unitary transformation of Hamiltonian can be performed as

$$\hat{H}_B = \hat{U}_B^{-1}\hat{H}_A\hat{U}_B - i\hbar\hat{U}_B^{-1}\frac{\partial \hat{U}_B}{\partial t}. \tag{18}$$

Then, after some algebra, we obtain the lastly transformed Hamiltonian in the form

$$\hat{H}_B(\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2, t) = \sum_{j=1}^{2} \hat{H}_{Bj}(\hat{q}_j, \hat{p}_j, t), \tag{19}$$

where

$$\hat{H}_{Bj}(\hat{q}_j, \hat{p}_j, t) = \frac{\hat{p}_j^2}{2\sqrt{L_1 L_2}} + \frac{1}{2}\sqrt{L_1 L_2}\,\omega_j^2 \hat{q}_j^2 + \mathcal{L}_{jp}(t), \tag{20}$$

with

$$\mathcal{L}_{jp}(t) = \frac{1}{2}\sqrt{L_1 L_2}\,\dot{q}_{jp}^2(t) - \frac{1}{2}\alpha_j q_{jp}^2(t). \tag{21}$$

Note that Eq. (20) is the same as that of the ordinary simple harmonic oscillator except for the last term.

3. WAVE FUNCTION

By expressing the Schrödinger equations related to Eq. (20) for each subscript j in the form

$$i\hbar \frac{\partial \psi_{1n}^{B}(q_1, t)}{\partial t} = H_{B1}\psi_{1n}^{B}(q_1, t), \tag{22}$$

$$i\hbar \frac{\partial \psi_{2m}^{B}(q_2, t)}{\partial t} = H_{B2}\psi_{2m}^{B}(q_2, t), \tag{23}$$

we can easily identify the wave function in the transformed system:

$$\psi_{n,m}^{B}(q_1, q_2, t) = \psi_{1n}^{B}(q_1, t)\psi_{2m}^{B}(q_2, t). \tag{24}$$

Here, ψ_{1n}^{B} and ψ_{2m}^{B} are

$$\psi_{1n}^{B}(q_1, t) = \left(\frac{\sqrt{L_1 L_2}\omega_1}{\hbar\pi}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\left[\left(\frac{\sqrt{L_1 L_2}\omega_1}{\hbar}\right)^{1/2} q_1\right] \times \exp\left(-\frac{\sqrt{L_1 L_2}\omega_1}{2\hbar} q_1^2\right) T_1(t), \tag{25}$$

$$\psi_{2m}^{B}(q_2, t) = \left(\frac{\sqrt{L_1 L_2}\omega_2}{\hbar\pi}\right)^{1/4} \frac{1}{\sqrt{2^m m!}} H_m\left[\left(\frac{\sqrt{L_1 L_2}\omega_2}{\hbar}\right)^{1/2} q_2\right] \times \exp\left(-\frac{\sqrt{L_1 L_2}\omega_2}{2\hbar} q_2^2\right) T_2(t), \tag{26}$$

where

$$T_1(t) = \exp\left[-i\omega_1 t\left(n + \frac{1}{2}\right) - \frac{i}{\hbar}\int_0^t \mathcal{L}_{1p}(t')\,\mathrm{d}t'\right], \tag{27}$$

$$T_2(t) = \exp\left[-i\omega_2 t\left(m + \frac{1}{2}\right) - \frac{i}{\hbar}\int_0^t \mathcal{L}_{2p}(t')\,\mathrm{d}t'\right]. \tag{28}$$

The wave function for the original system (untransformed system) can be derived from

$$\psi_{n,m}(q_1, q_2, t) = \hat{U}_A \hat{U}_B \psi_{n,m}^{B}(q_1, q_2, t). \tag{29}$$

Using Eqs. (4), (13), and (24), the above equation can be easily calculated as

$$\psi_{n,m}(q_1, q_2, t) = \phi_{n,m}(q_1, q_2, t)T(t), \tag{30}$$

where

$$\phi_{n,m}(q_1, q_2, t) = \left(\frac{L_1 L_2 \omega_1 \omega_2}{\hbar^2 \pi^2}\right)^{1/4} \frac{1}{\sqrt{2^{n+m} n! m!}} \exp\left[\frac{i}{\hbar}(p_{1p}(t) Q_1 + p_{2p}(t) Q_2)\right]$$

$$\times H_n \left[\left(\frac{\sqrt{L_1 L_2}\omega_1}{\hbar}\right)^{1/2} [Q_1 - q_{1p}(t)]\right] H_m \left[\left(\frac{\sqrt{L_1 L_2}\omega_2}{\hbar}\right)^{1/2} [Q_2 - q_{2p}(t)]\right]$$

$$\times \exp\left\{-\frac{\sqrt{L_1 L_2}}{2\hbar}\left[\omega_1 (Q_1 - q_{1p}(t))^2 + \omega_2 (Q_2 - q_{2p}(t))^2\right]\right\}, \quad (31)$$

$$T(t) = T_1(t) T_2(t), \quad (32)$$

with

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} (L_1/L_2)^{1/4} \cos\varphi & -(L_2/L_1)^{1/4} \sin\varphi \\ (L_1/L_2)^{1/4} \sin\varphi & (L_2/L_1)^{1/4} \cos\varphi \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}. \quad (33)$$

We now evaluate the wave function in p space. The Fourier transformation of Eq. (24) give

$$\bar{\psi}^B_{n,m}(p_1, p_2, t) = \bar{\psi}^B_{1n}(p_1, t) \bar{\psi}^B_{2m}(p_2, t), \quad (34)$$

where

$$\bar{\psi}^B_{1n}(p_1, t) = \left(\frac{1}{\sqrt{L_1 L_2}\omega_1 \hbar\pi}\right)^{1/4} \frac{(-i)^n}{\sqrt{2^n n!}} H_n \left[\left(\frac{1}{\sqrt{L_1 L_2}\omega_1 \hbar}\right)^{1/2} p_1\right]$$

$$\times \exp\left(-\frac{1}{2\sqrt{L_1 L_2}\omega_1 \hbar} p_1^2\right) T_1(t), \quad (35)$$

$$\bar{\psi}^B_{2m}(p_2, t) = \left(\frac{1}{\sqrt{L_1 L_2}\omega_2 \hbar\pi}\right)^{1/4} \frac{(-i)^m}{\sqrt{2^m m!}} H_m \left[\left(\frac{1}{\sqrt{L_1 L_2}\omega_2 \hbar}\right)^{1/2} p_2\right]$$

$$\times \exp\left(-\frac{1}{2\sqrt{L_1 L_2}\omega_2 \hbar} p_2^2\right) T_2(t). \quad (36)$$

Applying the similar procedure as that of q space, we derive the full wave function for the original system to be

$$\bar{\psi}_{n,m}(p_1, p_2, t) = \hat{U}_A \hat{U}_B \bar{\psi}^B_{n,m}(p_1, p_2, t)$$

$$= \bar{\phi}_{n,m}(p_1, p_2, t) T(t), \quad (37)$$

where

$$\begin{aligned}\bar{\phi}_{n,m}(p_1, p_2, t) = &\left(\frac{1}{L_1 L_2 \omega_1 \omega_2 \hbar^2 \pi^2}\right)^{1/4} \frac{(-i)^{n+m}}{\sqrt{2^{n+m} n! m!}} \\ &\times \exp\left[-\frac{i}{\hbar}[q_{1p}(t)(P_1 - p_{1p}(t)) + q_{2p}(t)(P_2 - p_{2p}(t))]\right] \\ &\times H_n\left[\left(\frac{1}{\sqrt{L_1 L_2}\omega_1 \hbar}\right)^{1/2} [P_1 - p_{1p}(t)]\right] \\ &\times H_m\left[\left(\frac{1}{\sqrt{L_1 L_2}\omega_2 \hbar}\right)^{1/2} [P_2 - p_{2p}(t)]\right] \\ &\times \exp\left\{-\frac{1}{2\hbar}\left[\frac{1}{\sqrt{L_1 L_2}\omega_1}(P_1 - p_{1p}(t))^2\right.\right. \\ &\left.\left.+\frac{1}{\sqrt{L_1 L_2}\omega_2}(P_2 - p_{2p}(t))^2\right]\right\}, \end{aligned} \tag{38}$$

with

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} (L_2/L_1)^{1/4}\cos\varphi & -(L_1/L_2)^{1/4}\sin\varphi \\ (L_2/L_1)^{1/4}\sin\varphi & (L_1/L_2)^{1/4}\cos\varphi \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}. \tag{39}$$

Equations (30) and (37) can be used to calculate various quantum mechanical expectation values. Using Eq. (30), the fluctuations for charges and currents can be calculated from

$$\Delta\hat{q}_j = \left[\langle\psi_{n,m}|\hat{q}_j^2|\psi_{n,m}\rangle - (\langle\psi_{n,m}|\hat{q}_j|\psi_{n,m}\rangle)^2\right]^{1/2}, \tag{40}$$

$$\Delta\hat{p}_j = \left[\langle\psi_{n,m}|\hat{p}_j^2|\psi_{n,m}\rangle - (\langle\psi_{n,m}|\hat{p}_j|\psi_{n,m}\rangle)^2\right]^{1/2}. \tag{41}$$

Then, we can easily confirm that the uncertainty products in Fock state are given by

$$\begin{aligned}\Delta\hat{q}_1\Delta\hat{p}_1 = &\frac{\hbar}{2}\left[(2n+1)^2\cos^4\varphi + (2m+1)^2\sin^4\varphi\right. \\ &\left.+\left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1}\right)(2n+1)(2m+1)\sin^2\varphi\cos^2\varphi\right]^{1/2}, \end{aligned} \tag{42}$$

$$\Delta\hat{q}_2\Delta\hat{p}_2 = \frac{\hbar}{2}\left[(2n+1)^2\sin^4\varphi + (2m+1)^2\cos^4\varphi \right.$$
$$\left. + \left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1}\right)(2n+1)(2m+1)\sin^2\varphi\cos^2\varphi\right]^{1/2}. \quad (43)$$

Note that the above two equations do not depend on $\varepsilon(t)$. It is known that the uncertainty product for the TDHS driven by time-dependent external forces such as electronic circuit with arbitrary power sources have nothing to do with external forces and the same as that of the system with no driving forces (Choi, 2004; Gweon and Choi, 2003). Therefore, these evaluations are well agree to the previous reports. For $n = m = 0$, Eqs. (42) and (43) becomes the same as that in Zhang *et al.* (2001) with $R_1 = R_2 = 0$.

4. THERMAL STATE

The thermal state may be conveniently described in terms of the Lewis–Riesenfeld (LR) invariant operator for the TDHS (Choi and Gweon, 2003; Ji and Kim, 1996). If the time-dependence of the TDHS disappear, LR invariant operator become the same as the Hamiltonian of the system. The invariant operator $\hat{I}_B$ for the transformed system can be easily constructed from

$$\frac{d\hat{I}_B}{dt} = \frac{\partial\hat{I}_B}{\partial t} + \frac{1}{i\hbar}[\hat{I}_B, \hat{H}_B] = 0. \quad (44)$$

Substitution of Eq. (19) into the above equation leads to

$$\hat{I}_B(\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2, t) = \sum_{j=1}^{2}\hat{I}_{Bj}(\hat{q}_j, \hat{p}_j, t), \quad (45)$$

where

$$\hat{I}_{Bj}(\hat{q}_j, \hat{p}_j, t) = \hbar\omega_j\left(\hat{a}_{Bj}^{\dagger}\hat{a}_{Bj} + \frac{1}{2}\right), \quad (46)$$

with annihilation operator $\hat{a}_{Bj}$ and creation operator $\hat{a}_{Bj}^{\dagger}$ that are given by

$$\hat{a}_{Bj} = \left(\frac{\sqrt{L_1L_2}\omega_j}{2\hbar}\right)^{1/2}\hat{q}_j + \frac{i}{(2\sqrt{L_1L_2}\omega_j\hbar)^{1/2}}\hat{p}_j, \quad (47)$$

$$\hat{a}_{Bj}^{\dagger} = \left(\frac{\sqrt{L_1L_2}\omega_j}{2\hbar}\right)^{1/2}\hat{q}_j - \frac{i}{(2\sqrt{L_1L_2}\omega_j\hbar)^{1/2}}\hat{p}_j. \quad (48)$$

The invariant operator for the original system can also be easily obtained from

$$\hat{I}(\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2, t) = \hat{U}_A \hat{U}_B \hat{I}_B(\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2, t) \hat{U}_B^{-1} \hat{U}_A^{-1}. \tag{49}$$

Therefore, we can write

$$\hat{I}(\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2, t) = \sum_{j=1}^{2} \hat{I}_j(\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2, t), \tag{50}$$

where

$$\hat{I}_j = \hbar\omega_j \left(\hat{a}_j^\dagger \hat{a}_j + \frac{1}{2} \right), \tag{51}$$

with

$$\hat{a}_j = \hat{U}_A \hat{U}_B \hat{a}_{Bj} \hat{U}_B^{-1} \hat{U}_A^{-1}, \tag{52}$$

$$\hat{a}_j^\dagger = \hat{U}_A \hat{U}_B \hat{a}_{Bj}^\dagger \hat{U}_B^{-1} \hat{U}_A^{-1}. \tag{53}$$

The ladder operators in the original system satisfies the boson commutation relation: $[\hat{a}_j, \hat{a}_j^\dagger] = 1$. For $\varepsilon(t) = 0$, the time-dependence of the Hamiltonian disappears and it can be shown that Eq. (50) become the same as the Hamiltonian of the system. When we represent the eigenvalue equation of the invariant operators in the form

$$\hat{I}_1 \phi_{1n}(q_1, q_2, t) = \lambda_{1n} \phi_{1n}(q_1, q_2, t), \tag{54}$$

$$\hat{I}_2 \phi_{2m}(q_1, q_2, t) = \lambda_{2m} \phi_{2m}(q_1, q_2, t), \tag{55}$$

the eigenvalues λ_{1n} and λ_{2m} can be written as

$$\lambda_{1n} = \hbar\omega_1 \left(n + \frac{1}{2} \right), \tag{56}$$

$$\lambda_{2m} = \hbar\omega_2 \left(m + \frac{1}{2} \right). \tag{57}$$

We see that the quantum number n and m are eigenvalues of $\hat{a}_1^\dagger \hat{a}_1$ and $\hat{a}_2^\dagger \hat{a}_2$, respectively. The relation between the eigenstates in Eqs. (54) and (55) and Eq. (31) is

$$\phi_{n,m}(q_1, q_2, t) = \phi_{1n}(q_1, q_2, t) \phi_{2m}(q_1, q_2, t). \tag{58}$$

Suppose that an ensemble of the oscillator particles is in thermal equilibrium with a temperature T and follows Bose-Einstein distribution statistics. Then, the density operator must be determined so as to satisfy the well-known Liouville–von Neumann equation (Ji and Kim, 1996) that is given by

$$\frac{\partial \hat{\rho}(t)}{\partial t} + \frac{1}{i\hbar} [\hat{\rho}(t),\ \hat{H}] = 0. \tag{59}$$

Since this equation is just the same as the relation written in terms of the invariant operator [see Eq. (44) for the transformed system], any function of the invariant operator may satisfy Eq. (59). Accordingly, the density matrix can be evaluated from

$$\rho(q_1, q_2, q_1', q_2', t) = \frac{1}{Z}\sum_{n,m=0}^{\infty} \psi_{n,m}(q_1, q_2, t) \exp\left\{ -\frac{\hbar}{kT}\left[\omega_1\left(n+\frac{1}{2}\right) + \omega_2\left(m+\frac{1}{2}\right)\right]\right\}\psi_{n,m}^{*}(q_1', q_2', t), \quad (60)$$

where Z is the partition function. The partition function is the sum of the Boltzmann factors over all states:

$$Z = \sum_{n,m=0}^{\infty} \left\langle \psi_{n,m}\middle| e^{-\hat{I}/(kT)} \middle|\psi_{n,m}\right\rangle. \quad (61)$$

After performing the summation, Z becomes simply

$$Z = \prod_{j=1}^{2} Z_j, \quad (62)$$

where

$$Z_j = \frac{1}{2\sinh[\hbar\omega_j/(2kT)]}. \quad (63)$$

Substituting Eq. (62) into Eq. (60) and after some algebra, we get the density matrix

$$\rho(q_1, q_2, q_1', q_2', t) = \prod_{j=1}^{2} \rho_j(q_1, q_2, q_1', q_2', t), \quad (64)$$

where

$$\rho_j = \left[\frac{\sqrt{L_1 L_2}\omega_j}{\hbar\pi}\tanh\left(\frac{\hbar\omega_j}{2kT}\right)\right]^{1/2} \exp\left[\frac{i}{\hbar}p_{jp}(t)(Q_j - Q_j')\right] \times \exp\left\{-\frac{\sqrt{L_1 L_2}\omega_j}{4\hbar}\left[[Q_j + Q_j' - 2q_{jp}(t)]^2 \tanh\left(\frac{\hbar\omega_j}{2kT}\right) + (Q_j - Q_j')^2\coth\left(\frac{\hbar\omega_j}{2kT}\right)\right]\right\}, \quad (65)$$

with

$$\begin{pmatrix} Q_1' \\ Q_2' \end{pmatrix} = \begin{pmatrix} (L_1/L_2)^{1/4}\cos\varphi & -(L_2/L_1)^{1/4}\sin\varphi \\ (L_1/L_2)^{1/4}\sin\varphi & (L_2/L_1)^{1/4}\cos\varphi \end{pmatrix} \begin{pmatrix} q_1' \\ q_2' \end{pmatrix}. \tag{66}$$

Similarly, we can calculate the density matrix in the momentum space by making use of Eq. (37) to be

$$\bar{\rho}(p_1, p_2, p_1', p_2', t) = \prod_{j=1}^{2} \bar{\rho}_j(p_1, p_2, p_1', p_2', t), \tag{67}$$

where

$$\begin{aligned} \bar{\rho}_j = &\left[\frac{1}{\sqrt{L_1L_2}\omega_j\,\hbar\pi}\tanh\left(\frac{\hbar\omega_j}{2kT}\right)\right]^{1/2}\exp\left[-\frac{i}{\hbar}q_{jp}(t)(P_j - P_j')\right] \\ &\times\exp\left\{-\frac{1}{4\,\hbar\sqrt{L_1L_2}\omega_j}\left[[P_j + P_j' - 2p_{jp}(t)]^2\tanh\left(\frac{\hbar\omega_j}{2kT}\right)\right.\right. \\ &\left.\left.+(P_j - P_j')^2\coth\left(\frac{\hbar\omega_j}{2kT}\right)\right]\right\}, \end{aligned} \tag{68}$$

with

$$\begin{pmatrix} P_1' \\ P_2' \end{pmatrix} = \begin{pmatrix} (L_2/L_1)^{1/4}\cos\varphi & -(L_1/L_2)^{1/4}\sin\varphi \\ (L_2/L_1)^{1/4}\sin\varphi & (L_1/L_2)^{1/4}\cos\varphi \end{pmatrix} \begin{pmatrix} p_1' \\ p_2' \end{pmatrix}. \tag{69}$$

5. UNCERTAINTY RELATION IN THERMAL STATE

In this section, we investigate the fluctuations of the canonical variables and uncertainty relation in thermal state. The diagonal element of Eqs. (64) and (67) are

$$f(q_1, q_2) = \prod_{j=1}^{2} f_j(q_1, q_2), \tag{70}$$

$$\bar{f}(p_1, p_2) = \prod_{j=1}^{2} \bar{f}_j(p_1, p_2), \tag{71}$$

where

$$f_j = \left[\frac{\sqrt{L_1 L_2}\omega_j}{\hbar\pi}\tanh\left(\frac{\hbar\omega_j}{2kT}\right)\right]^{1/2}$$
$$\times \exp\left[-\frac{\sqrt{L_1 L_2}\omega_j}{\hbar}\tanh\left(\frac{\hbar\omega_j}{2kT}\right)[Q_j - q_{jp}(t)]^2\right], \quad (72)$$

$$\bar{f}_j = \left[\frac{1}{\sqrt{L_1 L_2}\omega_j\,\hbar\pi}\tanh\left(\frac{\hbar\omega_j}{2kT}\right)\right]^{1/2}$$
$$\times \exp\left[-\frac{1}{\hbar\sqrt{L_1 L_2}\omega_j}\tanh\left(\frac{\hbar\omega_j}{2kT}\right)[P_j - p_{jp}(t)]^2\right]. \quad (73)$$

The expectation value of $\hat{q}_j^l$ and $\hat{p}_j^l$ in thermal state are

$$\langle \hat{q}_j^l \rangle_T = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q_j^l f(q_1, q_2)\,\mathrm{d}q_1\mathrm{d}q_2, \quad (74)$$

$$\langle \hat{p}_j^l \rangle_T = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_j^l \bar{f}(p_1, p_2)\mathrm{d}p_1\mathrm{d}p_2. \quad (75)$$

Using the integral formulas

$$\int_{-\infty}^{\infty} x\exp[-(ax^2 + bx + c)]\mathrm{d}x = -\frac{b}{2a}\sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2}{4a} - c\right), \quad (76)$$

$$\int_{-\infty}^{\infty} x^2\exp[-(ax^2 + bx + c)]\mathrm{d}x = \left(\frac{1}{2a} + \frac{b^2}{4a^2}\right)\sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2}{4a} - c\right), \quad (77)$$

Eq. (74) can be evaluated for $l = 1, 2$ to be

$$\langle \hat{q}_1 \rangle_T = \sqrt[4]{\frac{L_2}{L_1}}[q_{1p}(t)\cos\varphi + q_{2p}(t)\sin\varphi], \quad (78)$$

$$\langle \hat{q}_1^2 \rangle_T = \sqrt{\frac{L_2}{L_1}}\left\{\frac{\hbar}{2\sqrt{L_1 L_2}}\left[\frac{1}{\omega_1}\coth\left(\frac{\hbar\omega_1}{2kT}\right)\cos^2\varphi + \frac{1}{\omega_2}\coth\left(\frac{\hbar\omega_2}{2kT}\right)\sin^2\varphi\right]\right.$$
$$\left. + [q_{1p}(t)\cos\varphi + q_{2p}(t)\sin\varphi]^2\right\}, \quad (79)$$

$$\langle \hat{q}_2 \rangle_T = \sqrt[4]{\frac{L_1}{L_2}}[-q_{1p}(t)\sin\varphi + q_{2p}(t)\cos\varphi], \quad (80)$$

$$\langle \hat{q}_2^2 \rangle_T = \sqrt{\frac{L_1}{L_2}} \left\{ \frac{\hbar}{2\sqrt{L_1 L_2}} \left[\frac{1}{\omega_1} \coth\left(\frac{\hbar\omega_1}{2kT}\right) \sin^2\varphi + \frac{1}{\omega_2} \coth\left(\frac{\hbar\omega_2}{2kT}\right) \cos^2\varphi \right] + \left[-q_{1p}(t)\sin\varphi + q_{2p}(t)\cos\varphi \right]^2 \right\}. \tag{81}$$

By similar way, Eq. (75) with $l = 1, 2$ becomes

$$\langle \hat{p}_1 \rangle_T = \sqrt[4]{\frac{L_1}{L_2}} \left[p_{1p}(t)\cos\varphi + p_{2p}(t)\sin\varphi \right], \tag{82}$$

$$\langle \hat{p}_1^2 \rangle_T = \sqrt{\frac{L_1}{L_2}} \left\{ \frac{\sqrt{L_1 L_2}\,\hbar}{2} \left[\omega_1 \coth\left(\frac{\hbar\omega_1}{2kT}\right) \cos^2\varphi + \omega_2 \coth\left(\frac{\hbar\omega_2}{2kT}\right) \sin^2\varphi \right] + \left[p_{1p}(t)\cos\varphi + p_{2p}(t)\sin\varphi \right]^2 \right\}, \tag{83}$$

$$\langle \hat{p}_2 \rangle_T = \sqrt[4]{\frac{L_2}{L_1}} [-p_{1p}(t)\sin\varphi + p_{2p}(t)\cos\varphi], \tag{84}$$

$$\langle \hat{p}_2^2 \rangle_T = \sqrt{\frac{L_2}{L_1}} \left\{ \frac{\sqrt{L_1 L_2}\,\hbar}{2} \left[\omega_1 \coth\left(\frac{\hbar\omega_1}{2kT}\right) \sin^2\varphi + \omega_2 \coth\left(\frac{\hbar\omega_2}{2kT}\right) \cos^2\varphi \right] + [-p_{1p}(t)\sin\varphi + p_{2p}(t)\cos\varphi]^2 \right\}. \tag{85}$$

Therefore, from Eqs. (78)–(85), the fluctuations for charges and currents are

$$(\Delta \hat{q}_1)_T = \sqrt{\frac{\hbar}{2L_1\omega_1\omega_2}} \left[\omega_2 \coth\left(\frac{\hbar\omega_1}{2kT}\right) \cos^2\varphi + \omega_1 \coth\left(\frac{\hbar\omega_2}{2kT}\right) \sin^2\varphi \right]^{1/2}, \tag{86}$$

$$(\Delta \hat{q}_2)_T = \sqrt{\frac{\hbar}{2L_2\omega_1\omega_2}} \left[\omega_2 \coth\left(\frac{\hbar\omega_1}{2kT}\right) \sin^2\varphi + \omega_1 \coth\left(\frac{\hbar\omega_2}{2kT}\right) \cos^2\varphi \right]^{1/2}, \tag{87}$$

$$(\Delta \hat{p}_1)_T = \sqrt{\frac{L_1\,\hbar}{2}} \left[\omega_1 \coth\left(\frac{\hbar\omega_1}{2kT}\right) \cos^2\varphi + \omega_2 \coth\left(\frac{\hbar\omega_2}{2kT}\right) \sin^2\varphi \right]^{1/2}, \tag{88}$$

$$(\Delta \hat{p}_2)_T = \sqrt{\frac{L_2 \hbar}{2}} \left[\omega_1 \coth\left(\frac{\hbar\omega_1}{2kT}\right) \sin^2\varphi + \omega_2 \coth\left(\frac{\hbar\omega_2}{2kT}\right) \cos^2\varphi \right]^{1/2}. \tag{89}$$

Then, we can easily confirm that the uncertainty relations for charges and their conjugate currents are given by

$$(\Delta \hat{q}_1)_T(\Delta \hat{p}_1)_T = \frac{\hbar}{2}\left[\coth^2\left(\frac{\hbar\omega_1}{2kT}\right)\cos^4\varphi + \coth^2\left(\frac{\hbar\omega_2}{2kT}\right)\sin^4\varphi \right.$$
$$\left. + \left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1}\right) \coth\left(\frac{\hbar\omega_1}{2kT}\right) \coth\left(\frac{\hbar\omega_2}{2kT}\right) \sin^2\varphi\cos^2\varphi \right]^{1/2}, \tag{90}$$

$$(\Delta \hat{q}_2)_T(\Delta \hat{p}_2)_T = \frac{\hbar}{2}\left[\coth^2\left(\frac{\hbar\omega_1}{2kT}\right)\sin^4\varphi + \coth^2\left(\frac{\hbar\omega_2}{2kT}\right)\cos^4\varphi \right.$$
$$\left. + \left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1}\right) \coth\left(\frac{\hbar\omega_1}{2kT}\right) \coth\left(\frac{\hbar\omega_2}{2kT}\right) \sin^2\varphi\cos^2\varphi \right]^{1/2}. \tag{91}$$

If we replace from the above two equations as

$$\coth\left(\frac{\hbar\omega_1}{2kT}\right) \to 2n + 1, \tag{92}$$

$$\coth\left(\frac{\hbar\omega_2}{2kT}\right) \to 2m + 1, \tag{93}$$

the uncertainty relations in thermal state exactly recovers to that in Fock state, Eqs. (42) and (43). Since these uncertainty products always larger than $\hbar/2$, the uncertainty principle hold.

6. CONCLUSION

The Schrödinger equation for the mesoscopic capacitance coupled circuit with a power source is solved using two step unitary transformation. The original Hamiltonian Eq. (3) is somewhat complicate since it involves the cross term $\hat{q}_1\hat{q}_2$ and the power source term. In the first step transformation, we eliminated the cross term by introducing angle φ. The power source terms have been removed by means of the particular solutions of the firstly transformed Hamiltonian system in the second step transformation. Through these procedures, the original Hamiltonian converted to quite simple form, Eq. (19), so that we can easily treat it. The solution of the Schrödinger equation for the transformed Hamiltonian system is expressed as Eqs. (24) and (34) for q and p space, respectively. The relation for the wave function between transformed and original Hamiltonian system is Eq. (29). The exact full wave functions of the system are evaluated in Eqs. (30) and (37). These wave functions can be used to calculate various expectation values in Fock state.

By introducing the LR invariant operator Eq. (50), the thermal state of the system described. We supposed that an ensemble of oscillator particles is in thermal equilibrium with a temperature T and follows Bose–Einstein distribution statistics. The density operator must satisfy the Liouville–von Neumann equation. As $\varepsilon(t) \to 0$, the Hamiltonian of the system no longer depend on time and Eq. (50) becomes the same as the Hamiltonian. Then, the Boltzmann factor $e^{-\hat{I}/(kT)}$ in Eq. (61) recovers to $e^{-\hat{H}/(kT)}$. The density matrices in original system are Eqs. (64) and (67). By considering the diagonal elements of the density matrix we derived the fluctuations of charges and currents. These fluctuations do not depend on $\varepsilon(t)$. The uncertainty relations between charges and their conjugate currents are given by Eqs. (90) and (91) in thermal state. For $T \to 0$, they exactly recovers to that of Fock state, Eqs. (42) and (43), with $n, m = 0$. On the other hand, the uncertainty products increase with increasing temperature T. We confirmed that the uncertainty principle of the system always hold.

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